ference in magnitude between the minimum of J_z^E and the maximum of J_x^E (note the different scale for J_z^E and J_x^E).

The program as written also allows inclusion of lossy media by only a slight modification since ϵ_r, κ, and χ are already written as complex variables. Limitation in time and money restricted the number of results that could be obtained.

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Capacitance of a Circular Disk for Applications in Microwave Integrated Circuits

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***Abstract*—The quasi-static solution for a circular disk separated from a ground plane by a dielectric substrate is studied using the dual integral equation approach. A simple expression for equivalent capacitance is determined.**

INTRODUCTION

The analytical study of disk resonators is of considerable importance for applications in integrated circuits. In order to determine the resonant frequency of such structures, it becomes necessary to obtain the value of capacitance [1], [2]. Recently, the determination of capacitance for a circular disk resonator was accomplished using computer calculations based on a numerical approach in spectral domain [1], [3]. Although capacitance was determined readily, it appears that the determination of actual surface charge densities and potential functions may warrant inversion of matrices of large orders.

The main complication in such a class of problems arises because of the mixed boundary conditions involved. Various approaches have been put forth in the past to circumvent this complexity. Rikitake [4] used the relaxation method for studying electromagnetic induction in a plane sheet with a circular aperture. For a two-dimensional problem in Cartesian coordinates, use was made of conformal mapping [5]. A method using multiple partial images has been reported [6]. The capacitance of disk resonator in free space has also been obtained [7]. In this short paper, a convenient method for an accurate solution to the disk resonator is developed using dual integral equations [8]. A major advantage lies in the fact that capacitance, charge densities, and field functions are determined in terms of a quickly convergent series.

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FORMULATION

Consider the geometry shown in Fig. 1 for a circular disk resonator of radius "a," separated from a ground plane by a dielectric material. Without loss of generality, the radius is assumed to be unity. The disk is charged to potential V_0. The potential functions are considered to be $\phi_1(r,z)$ and $\phi_2(r,z)$ for $z > d$ and $0 < z < d$, respectively. Because of circular symmetry, the Hankel transforms of these functions may be defined as

$$\bar{\phi}_{1,2}(\alpha,z) = \int_0^\infty \phi_{1,2}(r,z) J_0(\alpha r) r \, dr. \tag{1}$$

Using the boundary conditions $\bar{\phi}_2(\alpha,0) = 0$ and $\bar{\phi}_1(\alpha, +\infty) = 0$, the following expressions for potentials are obtained:

$$\bar{\phi}_2(\alpha,z) = A(\alpha) \sinh \alpha z, \qquad 0 < z < d \tag{2}$$

$$\bar{\phi}_1(\alpha,z) = B(\alpha) \exp[-\alpha(z-d)], \qquad z > d. \tag{3}$$

The unknowns $A(\alpha)$ and $B(\alpha)$ are to be determined from the following boundary conditions. At the interface $z = d$,

$$\phi_1(r,d) = \phi_2(r,d). \tag{4}$$

In particular,

$$\phi_1(r,d) = \phi_2(r,d) = V_0, \qquad 0 < r < 1. \tag{5}$$

Also at $z = d$

$$\frac{\partial \phi_1(r,d)}{\partial z} - \epsilon_r \frac{\partial \phi_2(r,d)}{\partial z} = 0, \qquad r > 1. \tag{6}$$

Clearly from (4)

$$A(\alpha) \sinh \alpha d = B(\alpha) \tag{7}$$

and using (6) and (7), one can obtain the following dual integral equations

$$\int_0^\infty \frac{\alpha^{-1} \sinh \alpha d}{[\sinh \alpha d + \epsilon_r \cosh \alpha d]} \cdot f(\alpha) \cdot J_0(\alpha r) d\alpha = V_0, \qquad 0 < r < 1 \tag{8}$$

and

$$\int_0^\infty f(\alpha) \cdot J_0(\alpha r) d\alpha = 0, \qquad r > 1 \tag{9}$$

where

$$f(\alpha) = \alpha^2 A(\alpha)[\sinh \alpha d + \epsilon_r \cosh \alpha d]. \tag{10}$$

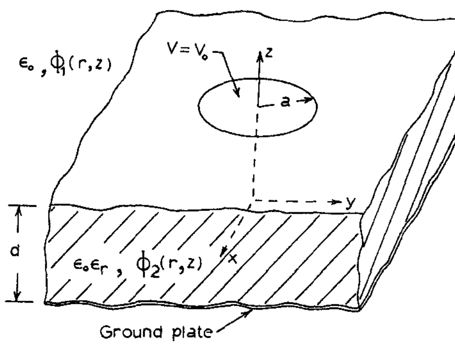

Fig. 1. Geometry of the problem.

Equations (8) and (9) are special cases ($\nu = 0$) of the general dual integral equations

$$\int_0^\infty G(p)f(p)J_\nu(rp)\,dp = g(r), \qquad 0 < r < 1 \tag{11}$$

$$\int_0^\infty f(p)J_\nu(rp)\,dp = 0, \qquad r > 1. \tag{12}$$

Such equations have been treated previously in the literature. Under certain conditions, these equations can be solved by employing Mellin transforms [9] or by converting them to Fredholm integral equations [10]. Here, use is made of a certain discontinuous property of integrals involving Bessel functions. The unknown function $f(\alpha)$ is assumed to be of the form

$$f(\alpha) = \alpha^{1-k}\sum_{m=0}^{\infty} a_m J_{2m+k}(\alpha) \tag{13}$$

where k is real, >0.

This automatically satisfies (9) and the general solution can be determined from (8). For the case $g(r) = A\cdot r^\nu$ applicable here, the solution is obtained as the following series [11, p. 115]:

$$a_n = \frac{2^{1-k}\Gamma(\nu+1)A}{\Gamma(\nu+k)}\cdot[\delta_n - c_n + c_n' - c_n'' + \cdots] \tag{14}$$

where

$$\delta_n = 0, \quad \text{for } n > 0; \qquad \delta_0 = 1$$

$$c_n = L_{0,n}; \qquad c_n' = \sum_{m=0}^{\infty} L_{m,n}c_n$$

$$c_n'' = \sum_{m=0}^{\infty} L_{m,n}c_n', \quad \text{etc.} \tag{15}$$

and

$$(2\nu + 4n + 2k)^{-1}L_{m,n} = \int_0^\infty \{p^{2-2k}G(p) - 1\}p^{-1}J_{\nu+2m+k}(p)\cdot J_{\nu+2n+k}(p)\,dp. \tag{16}$$

From practical considerations, k is chosen such that $\{p^{2-2k}G(p) - 1\}$ is as small as possible.

SOLUTION AND DISCUSSION

Initially, an attempt is made to obtain the solution for the condition $d \le 1$. To make $p^{2-2k}G(p)$ approach 1 for small values of d, (8) is rewritten as

$$\int_0^\infty G(\alpha)f(\alpha)J_0(\alpha r)\,d\alpha = \frac{\epsilon_r}{d}V_0 \tag{17}$$

where

$$G(\alpha) = \frac{\epsilon_r(\alpha d)^{-1}\tanh \alpha d}{\tanh \alpha d + \epsilon_r}. \tag{18}$$

Equation (18) is substituted in (16) and k is chosen to be "1." The coefficients are then obtained by performing the integrations in (16). Insertion in (14) yields the a_n. By retracing through the equations, $\bar{\phi}_{1,2}(\alpha,z)$ and hence the potential functions $\phi_{1,2}(r,z)$ can be determined.

In particular, if one is primarily interested in calculating the capacitance of the disk resonator, the total charge density on the disk is given by

$$\rho_{ST}(r,d) = \epsilon_0\sum_{m=0}^{\infty} a_m\int_0^\infty J_{2m+1}(\alpha)\cdot J_0(\alpha r)\,d\alpha \tag{19}$$

which is zero for $r > 1$ as expected. After interchanging the order of integrations and using an integral formula [12, p. 692], the total charge on the disk is obtained as

$$Q = 2\pi\epsilon_0\sum_{m=0}^{\infty} a_m\cdot\frac{\Gamma(1)\cdot\Gamma(m+1)}{2\Gamma(-m+1)\Gamma(m+2)\Gamma(m+1)}. \tag{20}$$

Since $\Gamma(-n) \to \infty$ for $n = 0,1,2,\cdots$, one gets

$$C = \frac{Q}{V_0} = \frac{\pi\epsilon_0}{V_0}\cdot a_0. \tag{21}$$

It may be interesting to note that the value of the total charge on the disk and hence the capacitance is uniquely determined by a_0 only while the charge density and potential functions are dependent on all the a_m. This implies that the transform of charge density should be of the form $(J_1(\alpha)/\alpha)$ for obtaining stationary value of capacitance [3].

The method followed for obtaining the capacitance for the case $d \ge a$ is similar to the aforementioned approach. In this case, (8) is rewritten as

$$\int_0^\infty G(\alpha)f(\alpha)J_0(\alpha r)\,d\alpha = (\epsilon_r + 1)V_0 \tag{22}$$

where

$$G(\alpha) = \frac{(\epsilon_r + 1)\alpha^{-1}\sinh \alpha d}{[\sinh \alpha d + \epsilon_r\cosh \alpha d]}. \tag{23}$$

Equation (23) is substituted in (16) and k is chosen to be $\frac{1}{2}$. In this case, the expression for capacitance is

$$C = \frac{2^{3/2}\pi^{1/2}\epsilon_0}{V_0}\cdot a_0. \tag{24}$$

As before, capacitance is determined by only one coefficient. The transform of charge density for numerical calculations is given by $\alpha^{-1/2}\cdot J_{1/2}(\alpha)$, which is the same as $(\sin \alpha/\alpha)$ obtained elsewhere [3].

The capacitance values can be calculated by completing the integrations in (16) and by substituting in (14) and (21) or (24). For $d \le a$, the integrations were performed on the digital computer and for one set of parameters d and ϵ_r, the computational time required was approximately 3 s on UNIVAC 1108. The integrand decreases asymptotically as the square of the argument and the integration was terminated at the upper limit of 30. Hence the capacitance values obtained for $d \le a$ are slightly lower than exact values. A more accurate determination of capacitance may require extending the upper limit of integration.

When $d \ge a$, the integration for $L_{m,n}$ can be obtained as a quickly convergent series in powers of $(1/d)$. This is indicated in the Appendix. It can be noted that as $d \to 0$, (14) and (21) show that the capacitance approaches $(\epsilon_r\epsilon_0\pi a^2/d)$ which is the value obtained when fringing fields are neglected. Similarly from (14), (24), and (A-3), for $d \to \infty$, the zeroth-order term or, in other words, the asymptotic value of $(Cd/\epsilon\pi a^2)$ is obtained as $(4(\epsilon_r + 1)d/\epsilon_r\pi a)$ which is $(8d/\pi a)$ for $\epsilon_r = 1$.

In Fig. 2, the normalized capacitance values are plotted as functions of distance d. The stationary values obtained elsewhere [3] are also indicated for comparison. Good agreement is evident. In Fig. 3, the resonant frequency using the equivalent value of static capacitance obtained here is plotted for $\epsilon_r = 2.65$. The curves by Mao *et al.* [13] and Itoh and Mittra [1] are also drawn. The experimental values reported in [1] are also shown. It can be noted that excellent agreement is obtained with the experimental results.

CONCLUDING REMARKS

The dual integral equation approach seems to be highly suitable for determining the potential function and capacitance of a circular disk resonator. No inversion of matrices is needed. The computational effort required is minimal. The complex situation of an arbi-

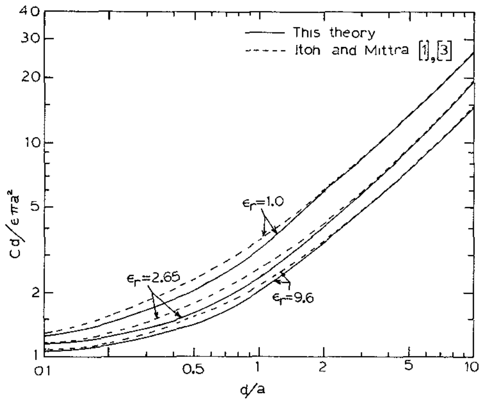

Fig. 2. Capacitance of a circular disk with dielectric substrate.

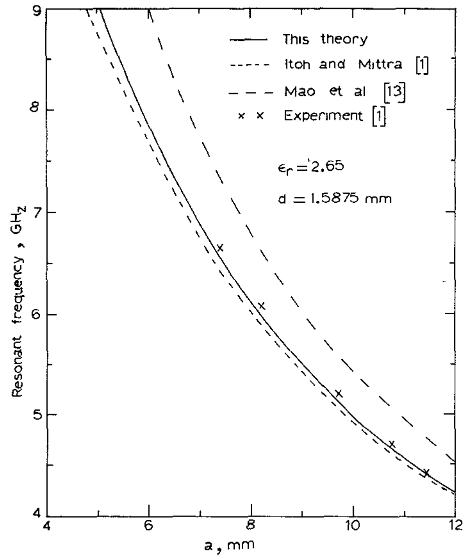

Fig. 3. Resonant frequency of disk resonator.

trary potential function on the disk can also be studied using this approach.

It may also be possible to extend the method to other problems involving mixed boundary conditions, e.g., striplines or microstrips [14] and circular strip resonators. For the latter case, the formulation would involve the discussion of triple integral equations. The solutions for these have been attempted in the literature [15].

APPENDIX

For $d \geq a$, the expression (16) for $L_{m,n}$ can be rewritten as

$$(4n+1)^{-1}L_{m,n} = \int_0^\infty \epsilon_r \cdot \frac{\sinh pd - \cosh pd}{\sinh pd + \epsilon_r \cosh pd} \cdot p^{-1} \cdot J_{2m+1/2}(p) \cdot J_{2n+1/2}(p)\, dp. \quad \text{(A-1)}$$

To obtain the result as a series in $(1/d)$, it is desirable to use the following expansion

$$\frac{\sinh pd - \cosh pd}{\sinh pd + \epsilon_r \cosh pd} = \frac{2}{\epsilon_r - 1}\sum_{s=1}^{\infty}(-1)^s\left(\frac{\epsilon_r - 1}{\epsilon_r + 1}\right)^s \exp(-2pds). \quad \text{(A-2)}$$

Using the formulas for Weber–Schafheitlin integral [16, p. 402], one gets

If needed, using the relation [17, p. 21]

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}x^n}{n^s} = \frac{1}{\Gamma(s)}\int_0^\infty u^{-s-1}\frac{x\cdot\exp(-u)}{1 + x\exp(-u)}\,du$$

and [12, p. 325]

$$\int_0^\infty \frac{u^{\nu-1}\exp(-\mu u)}{1 - \beta\exp(-u)}\,du = \Gamma(\nu)\Phi(\beta;\nu;\mu)$$

where $\Phi(\beta;\nu;\mu)$ is Lerch's exponent [18], the second series in (A-3) may be rewritten as

$$\sum_{s=1}^{\infty}(-1)^s\left(\frac{\epsilon_r-1}{\epsilon_r+1}\right)^s s^{-2(m+n+l)-1} = \left(\frac{\epsilon_r-1}{\epsilon_r+1}\right)\Phi\left[\left(-\frac{\epsilon_r-1}{\epsilon_r+1}\right);(2m+2n+2l+1);1\right].$$

ACKNOWLEDGMENT

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$$(4n+1)^{-1}L_{m,n} = \left(\frac{\epsilon_r}{\epsilon_r-1}\right)\sum_{l=0}^{\infty}\frac{(-1)^l\Gamma(2m+2n+2l+1)\cdot\Gamma(2m+2n+2l+2)}{2d(4d)^{2(m+n+l)}l!\,\Gamma(2m+l+\frac{3}{2})\cdot\Gamma(2n+l+\frac{3}{2})\cdot\Gamma(2m+2n+l+2)}\cdot\sum_{s=1}^{\infty}(-1)^s\left(\frac{\epsilon_r-1}{\epsilon_r+1}\right)^s s^{-2(m+n+l+1/2)}. \quad \text{(A-3)}$$

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A Note on Green's Function for Microstrip

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***Abstract*—The electrostatic Green's function for the open or covered microstrip line is obtained by an integral representation of the free space Green's function, the results of which may be applied to obtain approximately the characteristics of the lowest order "quasi-TEM" mode of microstrip.**

I. INTRODUCTION

The electrostatic Green's function for the open microstrip line may be obtained from extended image theory as illustrated by Silvester [1] and later by Weeks [2]. Weiss and Bryant [3] derived the covered microstrip Green's function by using a computer algorithm and this was refined by Farrar and Adams [4]. However, in [4] the Green's function is not explicitly given for all values of b/h[1] and the logarithmic singularity inherent in their series representation is not immediately apparent. A direct method of obtaining the Green's function for an open or covered microstrip without using extended image theory or a computer algorithm, valid for any b/h and retaining the logarithmic singularity, is now given. The approach used is conceptually similar to that of Kaden [5], in which the capacitance of a pair of circular cylindrical wires above a dielectric coated ground plane was determined.

II. FREE-SPACE GREEN'S FUNCTION

The free-space Green's function satisfying

$$\nabla^2\phi_0 = -\delta(x,y) \tag{1}$$

is [6]

$$\phi_0 = \frac{-1}{2\pi\epsilon_0}\log(r) \tag{2}$$

where

$$\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}, \qquad \delta(x,y)$$

is the two-dimensional Dirac delta function, ϵ_0 is the permittivity of free space, and $r = (x^2 + y^2)^{1/2}$. An integral representation of $\log(z)$ with $z = y + jx$ and $j^2 = -1$ is [7]

$$\log(z) = \int_0^\infty \frac{\exp(-\lambda) - \exp(-\lambda z)}{\lambda}\, d\lambda \tag{3}$$

where the integral converges provided $\mathrm{Re}(z) \geq 0$. On taking the real part of (3) and substituting for $\log(r)$ in (2), the free-space Green's function takes the form

$$\phi_0 = \frac{1}{2\pi\epsilon_0}\int_0^\infty \frac{\exp(-\lambda|y|)\cos(\lambda x) - \exp(-\lambda)}{\lambda}\, d\lambda. \tag{4}$$

Note that in (4) the x and y variables are separated so that derivatives may be easily obtained. Now the integral representation (4) is used to derive the electrostatic Green's function, the results of which may be applied to obtain approximately the characteristics of the lowest order "quasi-TEM" mode of a microstrip.

III. OPEN MICROSTRIP GREEN'S FUNCTION

To ϕ_0 are added functions satisfying the two-dimensional Laplace's equation, exhibiting the same x behavior as (4) and together with ϕ_0 satisfying the appropriate boundary conditions. Thus with reference to Fig. 1, the Green's function for the open microstrip may be chosen as

$$\phi_1(x,y) = \frac{1}{2\pi\epsilon_0} \cdot \int_0^\infty \frac{\exp(-\lambda|y-\Delta|) + f_1(\lambda)\exp[-\lambda(y-\Delta)]}{\lambda} \cdot \cos(\lambda x)\, d\lambda \tag{5}$$

for $y > 0$, and

$$\phi_2(x,y) = \frac{1}{2\pi\epsilon_0\epsilon_r} \cdot \int_0^\infty \frac{f_2(\lambda)\exp[-\lambda(y-\Delta)] + f_3(\lambda)\exp[\lambda(y-\Delta)]}{\lambda} \cdot \cos(\lambda x)\, d\lambda \tag{6}$$

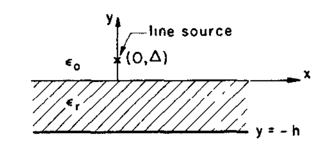

Fig. 1. Green's function geometry for the open microstrip.

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[1] b is the separation of the ground planes and h is the height of the line source from the bottom ground plane.